Exercise Sheet 1: The Homotopy Category

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1 The Homotopy Category

Definition 1 *Let $hTop$ denote the category whose objects are topological spaces and whose morphisms $X \to Y$ are homotopy classes of maps. We call $hTop$ the (classical)* ***homotopy category*** *of spaces.* $\square$

Thus the set of morphisms $X \to Y$ in $hTop$ is to be the set of equivalence classes of such maps under the relation of homotopy. We denote this set by

$$[X, Y]_0 = hTop(X, Y) = Top(X, Y)/\simeq .^1 \tag{1.1}$$

If $f : X \to Y$ is a map, it will be convenient to denote its homotopy class $[f] \in [X, Y]_0$.

Exercise 1.1 Show that $hTop$ is a category. What are the isomorphism classes of objects in $hTop$? $\square$

Let

$$Top \xrightarrow{\gamma} hTop \tag{1.2}$$

denote the functor which is the identity on objects and assigns to a map $f : X \to Y$ its homotopy class $[f] \in [X, Y]_0$.[2]

Exercise 1.2 Show that a map $f : X \to Y$ is a homotopy equivalence in Top if and only if $\gamma(f) = [f]$ is invertible in $hTop$. $\square$

We call a functor $Top \xrightarrow{F} \mathcal{C}$ which sends homotopy equivalences in Top to isomorphisms in $\mathcal{C}$ a **homotopy functor**.

Exercise 1.3 Show that the functor $Top \xrightarrow{\gamma} hTop$ is the universal homotopy functor: If $Top \xrightarrow{F} \mathcal{C}$ is a functor into a category $\mathcal{C}$ such that $F(f)$ is invertible in $\mathcal{C}$ whenever f is a homotopy equivalence in Top, then there is a unique functor $hTop \xrightarrow{\widehat{F}} \mathcal{C}$ such that $\widehat{F}\gamma = F$. $\square$

[1] The 0 is to indicate that we are working with *unpointed* maps and homotopies.

[2] Would you call γ a 'forgetful functor'?

Exercise 1.4 What implications does this have for the singular homology and cohomology functors? □

One thing suggested to us by Exercise 1.3 is that that we might like to think of $hTop$ as the category obtained from Top by *formally inverting* each homotopy equivalence. This is true, but not something we will dwell on here, since we do not yet have the tools to make such a formal inversion. At the moment I ask that you consider this statement only as motivation for some of the following.

Exercise 1.5

1. Show that if $Top \xrightarrow{F,G} \mathcal{C}$ are homotopy functors and $\alpha : F \Rightarrow G$ is a natural transformation, then there is a unique natural transformation $\widetilde{\alpha} : \widehat{F} \Rightarrow \widehat{G}$ such that $\widehat{\alpha}\gamma = \alpha$.

2. Show that if X is space, then $Y \mapsto [X, Y]_0$ defines a homotopy functor $Top \to Set$. Moreover that $f \mapsto [f]$ defines a natural transformation $Top(X, -) \Rightarrow [X, -]_0$[3]. Show that this is the universal natural transformation from $Top(X, -)$ into a homotopy functor.

3. What does *this* mean for the singular homology and cohomology functors? □

Next we will try to recover in $hTop$ a few constructions which are available in Top.

Exercise 1.6 Use the ideas of Exercise 1.4 to help show that there are bijections

$$[X \sqcup Y, Z]_0 \cong [X, Z]_0 \times [Y, Z]_0 \tag{1.3}$$

$$[X, Y \times Z]_0 \cong [X, Y]_0 \times [X, Z]_0 \tag{1.4}$$

natural in all three variables. Really what I am asking you to do here is to show that homotopy is compatible with disjoint unions and products, and to try to formalise this by making as few explicit point-set arguments as possible. □

Your solution to Exercise 1.6 tells you that the homotopy category has *finite* products and coproducts. With a little more work you will have proved the following.

Corollary 1.1 *$hTop$ has all products and coproducts.* ∎

Unfortunately the homotopy category has very few other limits and colimits. It does not have pullbacks or pushouts, for example. This is not obvious at this stage, but it is for this reason which we shall need to introduce *derived* versions of these construction at a later point. Despite this, here is one limit and one colimit which it is not difficult to spot:

Exercise 1.7 The homotopy category has an initial object and a terminal object. Which are they?

There is a notion weaker than that of homotopy functor which is also useful.

Definition 2 *A functor $Top \xrightarrow{F} Top$ is said to be* ***homotopical*** *if whenever $f : X \xrightarrow{\simeq} Y$ is a homotopy equivalence, then $F(f) : FX \to FY$ is a homotopy equivalence.* □

[3]It is technically more correct to write $[X, \gamma(-)]_0$ for the latter functor.

Exercise 1.8 Show that if F is a homotopical, then $Top \xrightarrow{\gamma F} hTop$ is a homotopy functor. $\square$

Exercise 1.9 For a fixed space X, show that the functors $Y \mapsto Y \times X$ and $Y \mapsto C(X, Y)$ are homotopical. $\square$

Exercise 1.10 Let Y be locally compact. Using Exercises 1.3 and 1.9 show that there are bijections

$$[X \times Y, Z_0] \cong [X, C(Y, Z)]_0 \tag{1.5}$$

which are natural in X, Z. $\square$